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Compressible Unsteady Vortex Lattice Method for Arbitrary Two-Dimensional Motion of Thin Profiles

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A numerical method based on the vortex methodology is presented for unsteady solution of the aerodynamics coefficients of a thin profile in subsonic and supersonic compressible flows. The numerical model is made through the airfoil discretization in uniform segments, and the singularity used is a vortex in compressible flow. Results for the proposed model are presented as the lift and pressure coefficients along the profile chord for some instants of time. The indicial response (unit step function) of the profile is obtained numerically. The method is also compared with solutions available in the literature.

Nomenclature

a_∞	=	sound speed, m/s
c_p	=	dimensionless pressure coefficient
M	=	dimensionless Mach number
t	=	time, s
U	=	uniform velocity, m/s
w	=	induced velocity, m/s
α	=	angle of attack, rad
Γ	=	vorticity, m^2/s
Δc_p	=	dimensionless pressure coefficient jump
$\delta\phi$	=	velocity potential jump, m^2/s
ϕ	=	velocity potential, m^2/s

Subscript

0	=	initial instant

Introduction

THIS paper reports on the study of the arbitrary motion of a two-dimensional thin airfoil in an otherwise subsonic or supersonic undisturbed stream. A simple and efficient numerical method based on point vortices is presented. The main limitation of the proposed procedure is that the mathematical modeling corresponds to the two-dimensional potential linearized equations. The basis of the proposed numerical procedure relies on the direct relationship between vortices and normal dipole panels with constant strength distribution.

It is well known that linearized compressible aerodynamics and linearized acoustic equations can be derived from one another. Therefore, if a Galilean transformation is performed between two reference systems, one fixed to the airfoil and the other fixed to the still air at infinity, the elementary classical wave-equation solutions from acoustics and the solutions of the convected wave equation can be obtained from the same transformation.

Extensive studies on elementary solutions for both wave and convected wave equations may be found in the literature [1–3].

If aerodynamic forces and moments on a thin profile are to be obtained for a given unsteady motion, in principle, there are three possible ways to solve the problem [4]. The first way is to employ the results from continuous harmonic time oscillations (obtained by standard techniques involving Fourier or Laplace transforms) to the subsonic, sonic, and supersonic linearized equations of the flow [1,2]. The second approach is to superimpose indicial aerodynamic responses by means of the superposition principle embodied in Duhamel's integral [2]. In this case the specified boundary conditions are satisfied both in the space and time domains. The third way consists of searching for the solution to the wave equation [4–6] or to the convected wave equation that fits the boundary conditions prescribed for the given motion [7]. It is precisely this last procedure that will be presented and discussed in the following paragraphs. Figure 1 illustrates the studied motion (i.e., a sudden step change in the angle of attack).

One way to obtain the results for an arbitrary motion based on the indicial response is the superposition procedure [2,8]. The given angle-of-attack variation is replaced by a number of small step changes. It is then apparent in the superposition process that the total lift at the desired time is equal to the sum of the increments of lift in each step of time. Another way to obtain an arbitrary motion is by changing the boundary conditions in the limits of linearized theory over the chord of the profile according to the desired motion [9]. This procedure of changing the boundary conditions to obtain an arbitrary motion has already been studied in an incompressible case [9].

The obtained results in this work are compared with Lomax et al. [10], who obtained theoretical results for the indicial lift-moment response for a step change in the angle of attack using an analogy between three-dimensional steady state and two-dimensional steady state [5,10]. The calculations are complex and allow theoretical values only for initial instants; other instants of the domain are completed with an exponential function. Thus, [10] is the major reference for comparing results when the subject is step-change indicial response in subsonic motion.

There are other ways in the literature to obtain the results for a step change in the angle of attack. Because the theoretical calculations are extremely complex, approximate functions that can prescribe the indicial response are studied. The studies of Beddoes [11] and Leishman [12] are classical, and the first is also used to compare our results for indicial lift in the subsonic case. Recently, due to computational advances, there were some results for the indicial motion obtained by using computational fluid dynamics (CFD) [13–16]. However, this approach has a very high computational cost compared with other methods (numerical, as proposed here, or approximate), taking about four orders of magnitude more computational time [13].

Mathematical Model

In a reference frame that translates steadily with uniform velocity U, the perturbation velocity potential due to an arbitrary small-amplitude motion of a thin airfoil and wake is governed by the following linear convected wave equation:

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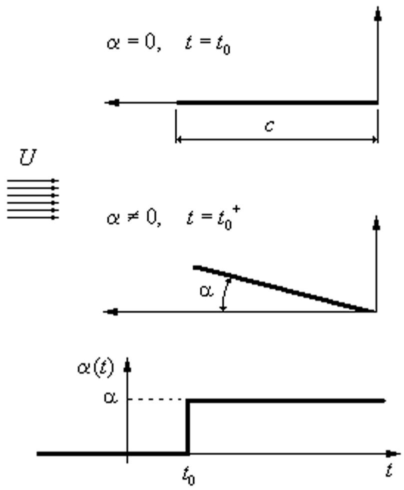

Fig. 1 Step change in the angle of attack.

$$(1 - M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{2M}{a_\infty}\frac{\partial^2\phi}{\partial x\partial t} - \frac{1}{a_\infty^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{1}$$

where a_∞ and M are, respectively, the undisturbed speed of sound and Mach number. The velocity U is in the positive x direction.

The condition of vanishing relative normal velocity on the airfoil, which lies on the $z = 0$ plane and spans over the x axis from the leading to the trailing edge, is written as

$$w_a(x, t) = \frac{\partial\phi}{\partial z} = \frac{\partial z_a}{\partial t} + U\frac{\partial z_a}{\partial x} \tag{2}$$

where z_a represents the instantaneous small transverse displacement of the mean camber line and the z axis points upward. To complete the mathematical model description, the pressure coefficient can be obtained from

$$c_p = -\frac{2}{U^2}\left(\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x}\right) \tag{3}$$

In subsonic flow, the Kutta condition must be imposed at the trailing edge. From Eq. (3), the pressure coefficient jump Δc_p on the airfoil and wake is obtained as

$$\Delta c_p = -\frac{2}{U^2}\left(\frac{\partial\delta\phi}{\partial t} + U\frac{\partial\delta\phi}{\partial x}\right) = -\frac{2}{U^2}\frac{D\delta\phi}{Dt} \tag{4}$$

where $\delta\phi$ stands for the velocity potential difference between the upper and lower sides. To assure pressure continuity over the trailing edge and wake, Eq. (4) is written as

$$\frac{D\delta\phi}{Dt} = 0 \tag{5}$$

Equation (5) guarantees that $\delta\phi$ at every point in the wake remains constant if displaced with the undisturbed velocity of the flow. This property is, in fact, a linearized version of Kelvin's theorem and will be employed in the proposed numerical model to assure that the Kutta and force-free wake conditions are enforced within the limits of the linearized theory.

Two solutions of Eq. (1) are employed to solve the problem at hand. The first solution corresponds to a point vortex of constant strength Γ, generated impulsively in an uniform flow at time t_0, which remains fixed at position x_0 (i.e., a bound vortex). The velocity induced by this point vortex along the x axis and $z = 0$ is given by

$$w(x, t) = -\frac{\Gamma}{2\pi a_\infty(t - t_0)}\frac{\sqrt{a_\infty^2(t - t_0)^2 - [(x - x_0) - U(t - t_0)]^2}}{(x - x_0)} \tag{6}$$

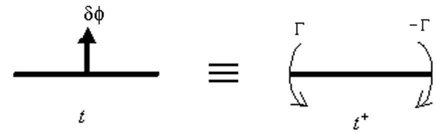

Fig. 2 Relationships existent between a vortex pair and the potential jump along the line that joins them.

The second solution is a free vortex that is convected by the flow with uniform velocity U, for which the inducing effect is

$$w(x, t) = -\frac{\Gamma}{2\pi a_\infty(t - t_0)}\frac{\sqrt{a_\infty^2(t - t_0)^2 - [(x - x_0) - U(t - t_0)]^2}}{(x - x_0) - U(t - t_0)} \tag{7}$$

and for which the solution is employed in the subsonic regime to satisfy the Kutta condition and to build in the free vortex sheet downstream of the trailing edge. If the flow is supersonic, only the real part of $w(x, t)$ must be taken into account for both bound and free vortices.

Solutions such as Eqs. (6) and (7) can be obtained in many ways. The most straightforward method is to take into account the formal analogy that exists between the two-dimensional wave equation, written for a fluid at rest, and the steady three-dimensional supersonic flow [2]. The semi-infinite line vortex that extends downstream in a steady supersonic flow is equivalent to the two-dimensional point vortex that spreads radially in a fluid at rest. Further applications of a Galilean transformation allow the final formulas, as in Eqs. (6) and (7). The well-known relationships existent between a vortex pair and the potential jump along the line that joins them are also verified in the unsteady compressible potential flow (see Fig. 2).

Numerical Model

The motion always begins from an impulsive start at $t = 0$ after the application of boundary conditions on the airfoil chord. The sequence of events of the proposed numerical procedure is summarized as follows.

Initially, the airfoil chord is divided in a convenient number of small and equally spaced panels. Over each panel, an unknown constant density normal dipole distribution is assumed. At $t = dt/2$, each dipole panel is replaced by its counter-rotating vortex pair placed at its edges. This arrangement automatically satisfies Kelvin's circulation theorem from the very start of motion. Taking into account the finiteness of the disturbance propagations, the induced velocities at control points (considering vortices of unitary strengths) are calculated from Eq. (6), and an influence coefficient matrix is constructed in an exact analogy to the classical vortex lattice method. After solving the linear system, for which the second member is composed of the boundary conditions, Eq. (2), the vortex intensities are obtained.

There are two main remarkable differences when comparing with the incompressible case. First, in the present case, each vortex does not affect all control points for a given time, but only those that are near it. In fact, the influence range is greater as the Mach number is lower. In the supersonic regime, only control points that are downstream of a given vortex are affected by it. This property is contained in elementary solutions given by Eqs. (6) and (7). The second difference is that for the terms that are located at the main diagonal of the influence coefficient matrix, an additional term taken from piston theory [2],

$$w(x, \delta\phi = 1) = \frac{M}{2U^2\,dt} \tag{8}$$

must be added in an analogy to what is done in supersonic vortex lattice method [17].

During the elapsed time interval dt, the trailing-edge bound vortex becomes free in subsonic flow and is shed with undisturbed flow velocity. This mechanism satisfies the Kutta condition.

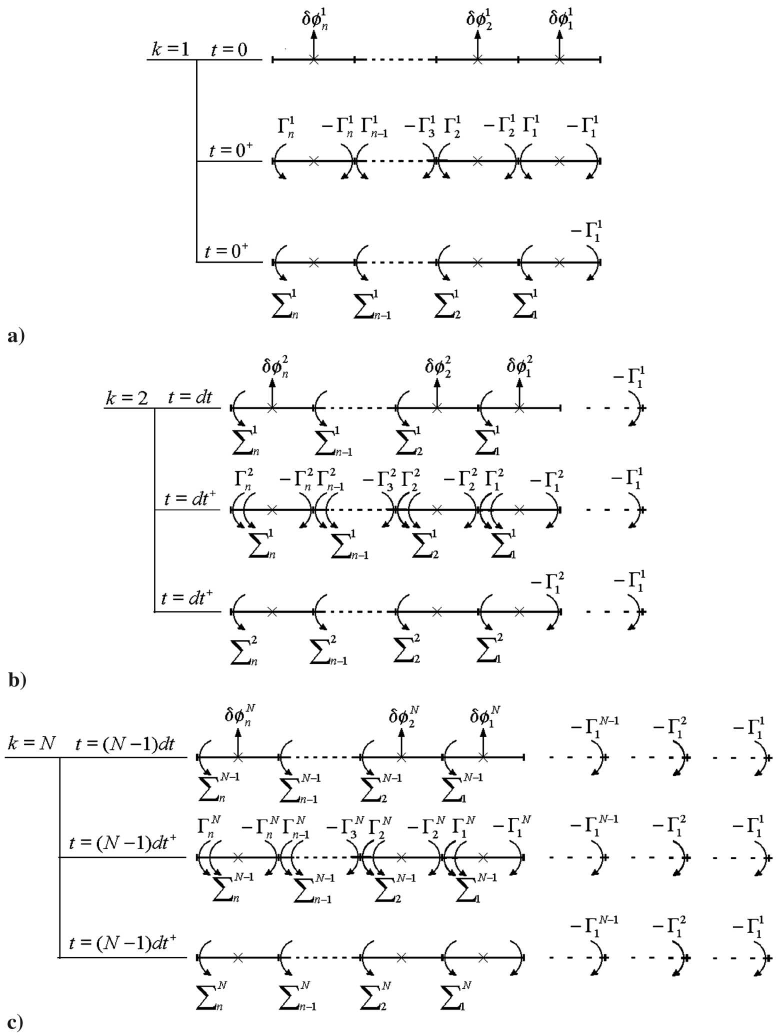

Fig. 3 Numerical scheme for subsonic case: a) $k = 1$, b) $k = 2$, and c) $k = N$.

At the next time step $t = (dt/2) + dt$, a new constant density normal dipole distribution is superimposed on the former vortex system, which is again replaced by a counter-rotating vortices pair, and boundary conditions are enforced again. The newly created vortex at the trailing edge in subsonic flow is shed, as was the older one during time interval dt. This procedure generates a free wake in the form of a row of adjacent discrete vortices. In supersonic flow, all vortices are tied to the airfoil. Although vorticity might be shed, if circulation changes over the airfoil, its influence is then zero, because all control points are situated upstream of the shed wake.

After each time step, the pressure-jump distribution, lifting force, and pitching moment coefficients can be calculated as functions of time from direct application of Eq. (4).

The numerical scheme is shown in Fig. 3 for the subsonic motion. It is possible to visualize the same time domain by considering the time as an axis. This scheme permits clearer understanding of the similarity to the vortex lattice method and is shown in Fig. 4.

Results

The proposed numerical method was employed to obtain the load distribution and the lift coefficient time variation for a sudden change of the angle of attack of a flat plate in an otherwise steady uniform flow. This problem is, in fact, a very demanding one for any numerical method and is the only well-documented case that can be used to validate a scheme specifically designed to treat an arbitrary motion.

In Fig. 5, the load distribution per angle of attack in subsonic flow is shown for three different instants of time. The singularity at the leading edge is apparent even for $Ut/c = 0.2$, that is, at the very start of the process. Also apparent is the vanishing of the load at the trailing edge. The overall agreement with the Lomax data [10] is quite good. In Fig. 6, the time evolution of the lift coefficient is presented for two subsonic Mach numbers. The starting range, roughly until the plate moves two chords, is remarkably well-described by the present numerical method.

The supersonic solutions are also presented in Figs. 7 and 8. In Fig. 7, for the least instant of time, the rear part of the flat plate is still under the influence of the uniform load from time $t > 0$, whereas the front end is already caught by waves generated from the leading edge. On the other hand, after 1.5 Ut/c, most of the profile chord

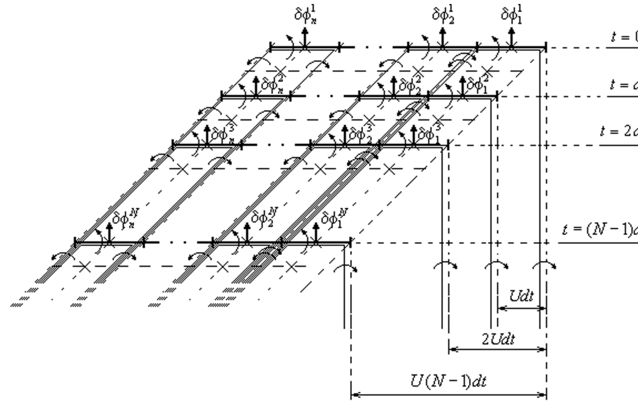

Fig. 4 Numerical scheme considering time as an axis showing the vortex lattice appearance.

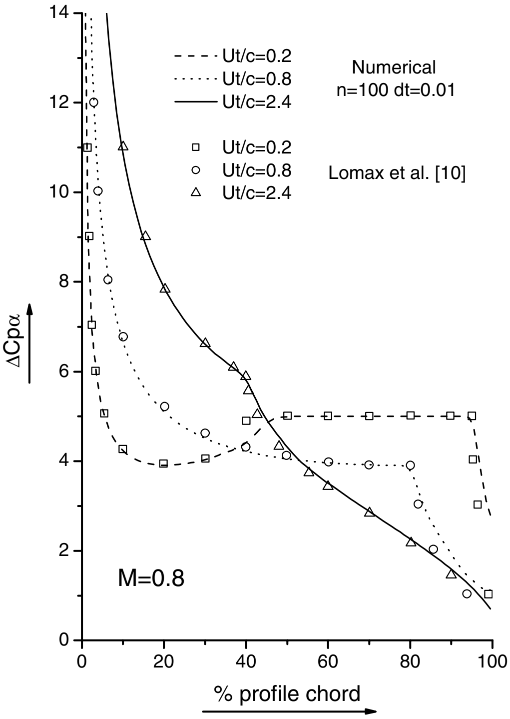

Fig. 5 Variation of two-dimensional indicial load distribution per radian with percent chord; Mach number $M = 0.8$, chord discretization $n = 100$, and time step $dt = 0.01$.

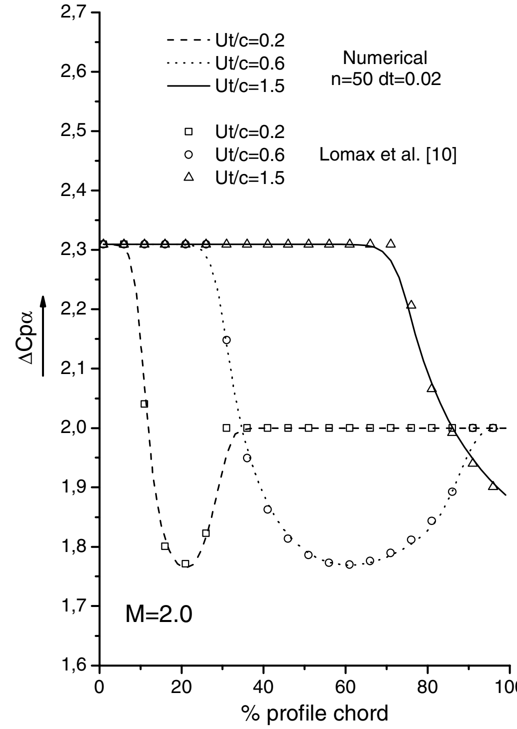

Fig. 7 Variation of two-dimensional indicial load distribution per radian with percent chord; $M = 2.0$, $n = 50$, and $dt = 0.02$.

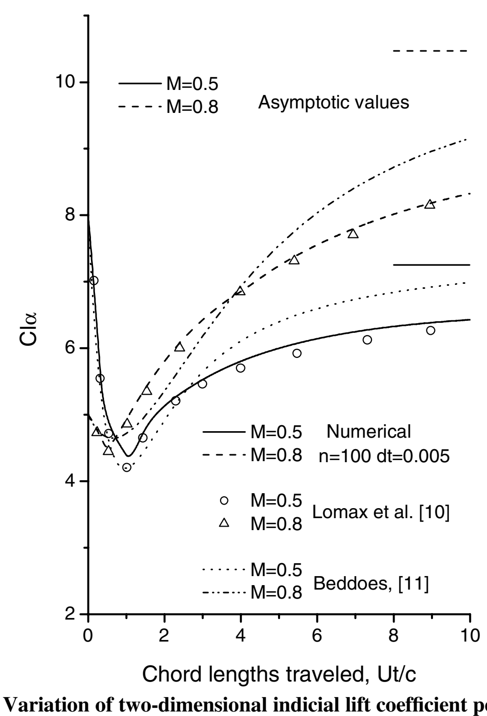

Fig. 6 Variation of two-dimensional indicial lift coefficient per radian with chord lengths traveled, for subsonic motion; $n = 100$ and $dt = 0.005$.

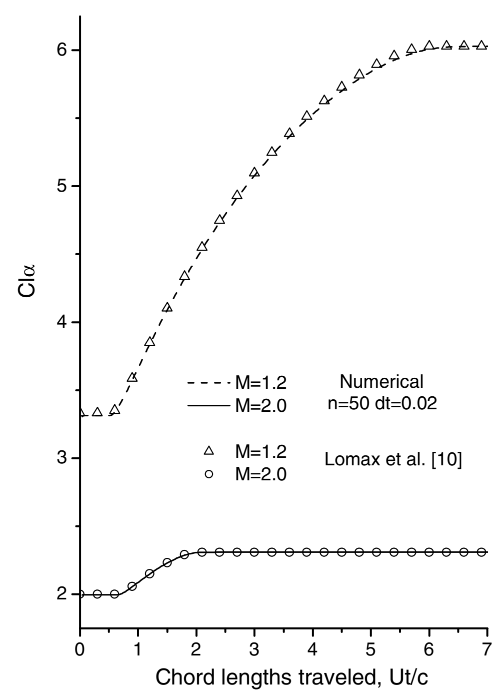

Fig. 8 Variation of two-dimensional indicial lift coefficient per radian with chord lengths traveled for supersonic Mach numbers; $n = 50$ and $dt = 0.02$.

load has already reached the steady-state value. In Fig. 8, two supersonic solutions are compared with the analytical solutions [10]. The reader can observe that the distributions are practically coincident.

In Figs. 9 and 10, the load distribution per angle of attack in subsonic and supersonic flows is shown along the time traveled. It is remarkable that the same characteristics are observed in Figs. 5 and 7, although in a continuous way, along the time traveled. In subsonic flow, Fig. 9 shows the variation of two-dimensional indicial load distribution per radian along the traveled time for $M = 0.5$. Note that there is a predominant nonsteady region ($Ut/c < 1$) followed by a well-defined standard region ($Ut/c > 1$) in which only the growing circulating portion exists and the load distribution already appears as a steady flow. In the supersonic case, Fig. 10 shows the variation of two-dimensional indicial load distribution per radian along the traveled time for $M = 2.0$. It is evident that the three typical load curves are the same as those already observed in Fig. 7 and the flow achieves steady flow at $Ut/c = M/(M - 1)$.

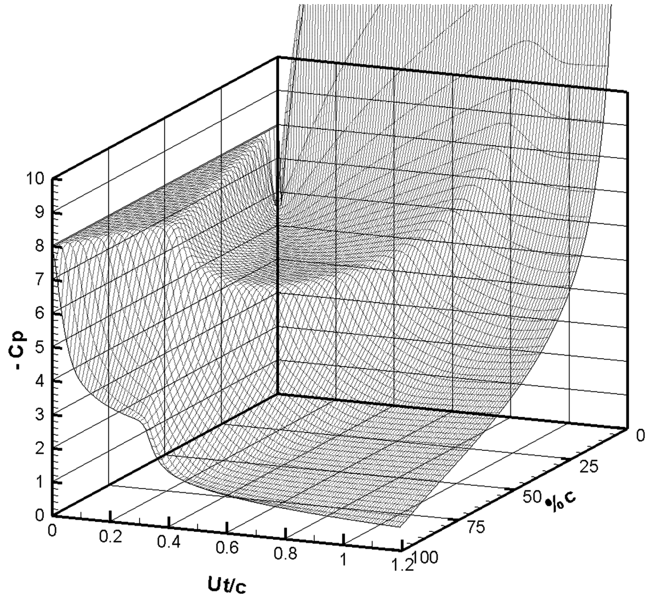

Fig. 9 Variation of two-dimensional indicial load distribution per radian along the traveled time; $M = 0.5$, $n = 50$, and $dt = 0.01$.

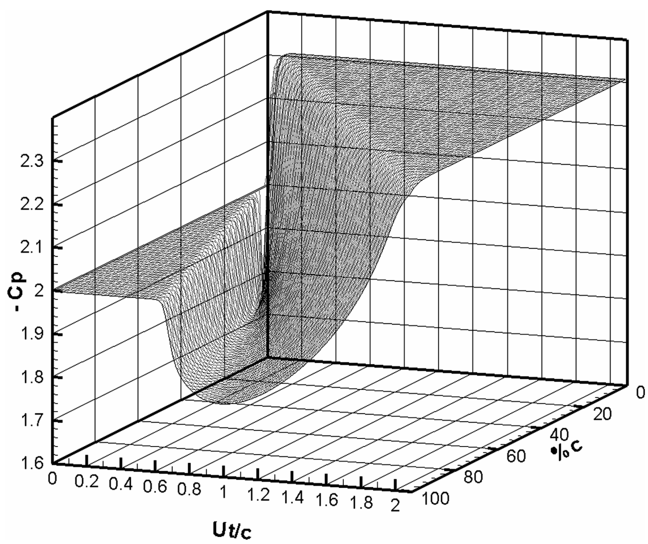

Fig. 10 Variation of two-dimensional indicial load distribution per radian along the traveled time; $M = 2.0$, $n = 50$, and $dt = 0.02$.

Conclusions

The numerical method presented here is, in fact, the natural extension of the compressible regime of the classical vortex lattice method in its two-dimensional version.

The main physical difference is the finiteness of the disturbances' propagations. The correspondence between vortex and normal dipole panels with constant density, as well as the concepts of bound and free vortices, remains valid in both subsonic and supersonic regimes and is an essential feature of the proposed numerical scheme.

The vantage of this method when compared with others is that it is possible to calculate the loads and forces over the profile and over the profile neighborhood too, such as a CFD calculation, at a low computational cost. And in the approximate methods based on exponential functions, it is not possible to obtain the loads over the profile, only the forces.

The proposed method is the first to use a simple numerical scheme to obtain forces and loads for an arbitrary motion of a profile in the unsteady compressible domain. To validate the method, a step change of the angle of attack (an indicial motion) was calculated. An arbitrary motion can be obtained using a superposition process or simply by changing the boundary conditions, because it is a numerical scheme. Using this method, it is possible to obtain results for many important motions in applied aerodynamics, such as sharp-edged gust and airfoil–vortex interaction, which is very important for the study of the helicopter noise and cosine gust used in the development of the actual civil aircraft.

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